International Journal of Theoretical Physics, Vol. 25, No. 11, 1986

Boson–Fermion Duality in Four Dimensions: Comments on the Paper of Luther and Schotte

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Received June 18, 1986

The Fock space for the fermion field can be identified with the Fock space for the boson field, provided the overall numbers of internal degrees of freedom are the same. As a consequence, the respective free field Hamiltonian systems are equivalent (dual): the four-component neutrino model is thus equivalent to the doublet of independent ("electric" and "magnetic," respectively) Maxwell fields, which are quantized in the Coulomb gauge. This statement arises on the field theory level, and seems to make doubtful the claim that realistic photons can be constructed from (bound) neutrino pairs: each (anti)neutrino degree should be represented by the photon-type ("electric" and "magnetic," respectively) degree of freedom.

1. MOTIVATION

An old problem of the neutrino theory of light has been recently revived (Aratyn, 1983; Luther and Schotte, 1984; Mickelsson, 1985) and even pursued (Luther and Schotte, 1984) to the notion of photon–neutrino duality [the subtitle of Luther and Schotte (1984) states: neutrinos from photons and vice versa], which is a culmination of the approach of Bloch, Jordan, Kronig, Haldane, and Luther (and others) to the description of Fermi systems in terms of bosons.

The construction of Luther and Schotte (1984) refers to momentum space lattices (periodic finite box in configuration space) and thus the boson or fermion creation (annihilation) operators are related to discrete physical degrees of freedom (modes). A serious problem whose analysis in Luther and Schotte (1984) is not very detailed is connected with the infinite-volume

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0020-7748/86/1100-1193$05.00/0

limit (i.e., continuous mode distribution). The trouble seems to arise from the very recent demonstration of the boson and fermion Fock space unification (see, e.g., Garbaczewski, 1985a, but also Garbaczewski, 1985b).

A much less important point is that the photon-neutrino duality of Luther and Schotte results in splitting the bosons into two independent potentials. One can be identified with the standard ("electric") Maxwell potential, while the other is not the dual one in the well-defined Weinberg (1965) sense. Hence, its relationship to electromagnetism is not that obvious.

Coming back to the above-mentioned Fock space unification issue, let us emphasize that in Aratyn (1983) and Luther and Schotte (1984) the Fock representations of the CCR and CAR algebra are used, and for the notion of duality it is essential that the vacuum state is common for both. Some time ago a general theorem was proved on the relationship of such representations (Garbaczewski and Rzewuski, 1974; Garbaczewski, 1975). Quite recently it has been strengthened by the boson-fermion Fock space unification (rigorous) proof (Hudson and Parthasarathy, 1986; Garbaczewski, 1985a). Since the field theory case is finally what is of interest in Luther and Schotte (1984), the conclusions about photons as neutrino composites must not be inconsistent with Hudson and Parthasarathy (1986) and Garbaczewski (1985a).

Our anlysis from the very beginning will be performed on the field theory level, and its starting point will be the second quantized version of the four-component neutrino theory. It is to be contrasted with Luther and Schotte (1984), where the whole of the analysis was made for the model in a finite box, hence corresponding to the discrete momentum distribution.

The boson-fermion relationship we have in mind from the start is very different from the traditional "fusion" route, since each single fermion degree of freedom will find its respective boson image and conversely. In particular, if we apply it to field theory models in Fock space (the space-time dimension or spin choice is immaterial here), we arrive at the following observation (Garbaczewski, 1985a):

Each local Fermi field theory model, if quantized by means of the Fock representation, admits an equivalent pure boson realization (fields may violate the local commutativity condition on the boson level). If adapted to the standard Fermi field Hamiltonian in its diagonal form of, say (Bjorken and Drell, 1965; Lee, 1981),

$$
\begin{gathered}
H_{\mathrm{F}}=\sum_{s=\pm}\int d^3p\, E_p[b^*(p,s)b(p,s)+d^*(p,s)d(p,s)] \\
[b(p,s),b^*(p',s')]_+=\delta_{ss'}\delta(p-p')=[d(p,s),d^*(p',s')]_+ \qquad (1) \\
b(p,s)|0)=d(p,s)|0)=0 \qquad \forall p,s
\end{gathered}
$$

the construction of Garbaczewski (1974, 1975) automatically leads to the conclusion that in the very same Fock space $\mathscr{H}$ [in connection with the Fock space identification issue see, e.g., Hudson and Parthasarathy (1986) and Garbaczewski (1985a)] we deal as well with the Bose system

$$
\begin{gathered}
H_{\mathrm{B}}=\sum_{s=\pm}\int d^3p\,E_p[B^*(p,s)B(p,s)+D^*(p,s)D(p,s)] \\
[B(p,s),B^*(p',s')]_-\equiv\delta_{ss'}\delta(p-p')\equiv[D(p,s),D^*(p',s')]_- \\
B(p,s)|0)=0=D(p,s)|0),\qquad \forall p,s \\
B^*(p,s)|0)=b^*(p,s)|0),\qquad D^*(p,s)|0)=d^*(p,s)|0)
\end{gathered}
\tag{2}
$$

which is equivalent to the previous Fermi system in the following sense:

$$
\begin{gathered}
H_{\mathrm{B}}|\psi)=H_{\mathrm{F}}|\psi)\qquad \forall|\psi)\in\mathscr{D}\subset\mathscr{H} \\
\exp(iH_{\mathrm{B}}t)|\psi)=\exp(iH_{\mathrm{F}}t)|\psi)\qquad \forall|\psi)\in\mathscr{H}
\end{gathered}
\tag{3}
$$

Let us emphasize that in the above we deal with diagonal Hamiltonians and that in the nondiagonal case the fermion partner obeying (3) cannot be found for all boson models.

Also see, e.g., the detailed study of analogous phenomena for interacting models in $1+1$ dimensions (Garbaczewski, 1983, 1984).

If we specialize E_p to the case of mass-zero particles $E_p=|p|$, the identities (3) provide us with the most general solution to the problem of representing the four-component neutrino Hamiltonian in the Hilbert space of the Bose system. The reverse problem of representing the Bose Hamiltonian in the fermion Hilbert space is solved by (3) as well.

It is *a priori* impossible to reconcile the ansatz of "fusion" = neutrino theory of light (fermion pairs superposing to a boson) with (3), where each fermion degree of freedom finds its respective boson analog. Though this ansatz is motivated by the need to reconcile the spin-$\frac{1}{2}$ and spin-1 transformation properties of one-particle states, it is inconsistent with the continuum field theory identities (3), whose origin is investigated in Section 2.

Since the identities (3) hold unambiguously, it is our aim to understand the physical meaning of the related boson degrees of freedom: in fact, we are strongly motivated by the paper by Luther and Schotte, so that we expect them to be related to electromagnetism.

At this point we turn to the classic papers (Weinberg, 1964a-c, 1965) on the transformation properties of fields and states as well as on the construction of relativistic fields while a physical (helicity) information about particle states is supplemented by the spin specification. The analysis of the quantization of gauge fields and of the role of the weak local commutativity (Strocchi, 1967, 1970) is extremely useful in this respect. We were struck by the fact that the monograph by Itzykson and Zuber has no description of the results of Weinberg's and Strocchi's papers in application to massless fields, and the Gupta–Bleuler formalism is left as the only candidate. The main results of this paper are listed in the conclusions.

2. FOCK SPACE UNIFICATION FOR FERMIONS AND BOSONS

We outline the aspects of Garbaczewski (1974, 1975, 1985a) and Hudson and Parthasarathy (1986) that underly the identities (3). One must be warned that the situation becomes more complicated if we pass to interacting fields: then it is not true that all boson models allow for a pure fermion reconstruction, while the converse still retains its validity (see, e.g., Garbaczewski, 1985a). Any construction of fermion (CAR) algebra generators falls into the framework established in Garbaczewski (1974, 1975). Namely, we have shown that certain homomorphisms of the nth power space $K^{\otimes n}$ can be exploited to produce Fock representations of the CAR algebra in the Fock space of this for the CCR (boson) algebra. Effectively, the fermions-from-bosons program was accomplished there. We have the Fock space $\mathscr{H}$ carrying the representation of the CCR algebra:

$$\begin{gathered}[b(f), b(g)^*]_- = \langle f, g\rangle 1_B \\ [b(f), b(g)]_- = 0 \\ b(f)|0) = 0 \qquad \forall f \in K\end{gathered} \tag{4}$$

where K is the test function space and $\langle f, g\rangle$ denotes the scalar product in K. The analysis of Garbaczewski (1974, 1985) demonstrates that in $\mathscr{H}$ a Fock representation of the CAR (fermion) algebra is induced such that its generators obey

$$\begin{gathered}[a(f), a(g)^*]_+ = \langle f, g\rangle 1_F \\ [a(f), a(g)]_+ = 0 \\ a(f)|0) = 0 \qquad \forall f \in K, \qquad a(f)^*|0) = b(f)^*|0)\end{gathered} \tag{5}$$

In pysical terms this means that the Fock vacuum is common for both representations, and that the overall number of internal degrees of freedom for fermions and bosons is the same (and is determined by the choice of K). Let $K = \bigoplus_1^N L^2(R^3)$, which corresponds to the number N of internal degrees of freedom. On the basis of Garbaczewski (1974, 1975) (compare, e.g., also Garbaczewski, 1984, 1985a, b), it is easy to check that each n-particle fermion state in the boson Fock space has the following realization (the homomorphism of $K^{\otimes n}$ must be specified accordingly):

$$\begin{aligned}
&a(f_1)^* \cdots a(f_n)^*|0) \\
&= \sum_{\alpha_1=1}^{N} \cdots \sum_{\alpha_n=1}^{N} \int d^3k_1 \cdots d^3k_n f_1^{\alpha_1}(k_1) \cdots f_n^{\alpha_n}(k_n) \\
&a^*_{\alpha_1}(k_1) \cdots a^*_{\alpha_n}(k_n)|0) \\
&= \sum_{1}^{N} \int d^3k_1 \cdots \int d^3k_n f_1^{\alpha_1}(k_1) \cdots f_n^{\alpha_n}(k_n) \\
&\quad \times \sigma(\alpha_1, k_1; \alpha_2, k_2; \ldots; \alpha_n, k_n) b^*_{\alpha_1}(k_1) \cdots b^*_{\alpha_n}(k_n)|0)
\end{aligned} \tag{6}$$

where, obviously

$$\begin{aligned}
&[b_\alpha(k), b^*_\beta(p)]_- \equiv \delta_{\alpha\beta}\delta(k-p)1_{\mathrm{B}} \\
&[a_\alpha(k), a^*_\beta(p)]_+ \equiv \delta_{\alpha\beta}\delta(k-p)1_{\mathrm{F}} \\
&\quad a_\alpha(k)|0) = b_\alpha(k)|0) = 0 \qquad \forall \alpha, k \\
&\quad a^*_\alpha(k)|0) = b^*_\alpha(k)|0) \\
&\quad a^*_{\alpha_1}(k_1) \cdots a^*_{\alpha_n}(k_n)|0) \\
&\qquad = \sigma(\alpha_1, k_1; \ldots; \alpha_n, k_n) b^*_{\alpha_1}(k_1) \cdots b^*_{\alpha_n}(k_n)|0)
\end{aligned} \tag{7}$$

In the above, $\sigma(\alpha_1, k_1; \ldots; \alpha_n, k_n)$ is the Friedrichs-Klauder ordering symbol, taking the value 0 or ± 1:

$$\begin{aligned}
&\sigma(\alpha_{\pi(1)}, k_{\pi(1)}; \ldots; \alpha_{\pi(n)}, k_{\pi(n)} = (-1)^\pi \\
&\qquad (\alpha, k)_i \neq (\alpha, k)_j \qquad \forall i, j
\end{aligned} \tag{8}$$

The symbol vanishes if any pair (α, k) of labels appears more than once. To unambiguously define the meaning of the permutation π we adopt the convention of linear ordering: $(\alpha, k) < (\alpha', k')$ if $\alpha < \alpha'$ or $\alpha = \alpha'$, $k_1 < k_1'$, or $\alpha = \alpha'$, $k_{(1)} = k'_{(1)}$, $k_{(2)} < k'_{(2)}$, or $\alpha = \alpha'$, $k_{(1)} = k'_{(1)}$, $k_{(2)} = k'_{(2)}$, $k_{(3)} < k'_{(3)}$, $k = \{k_{(1)}, k_{(2)}, k_{(3)}\} \in R^3$. Quite analogously to (6), the n-particle boson state in $\mathscr{H}$ can be given both in terms of boson and fermion generators:

$$\begin{aligned}
&b(f_1)^* \cdots b(f_n)^*|0) \\
&= \sum_{\alpha_1 \cdots \alpha_n}^{N} \int d^3k_1 \cdots \int d^3k_n f_1^{\alpha_1}(k_1) \cdots f_n^{\alpha_n}(k_n) b^*_{\alpha_1}(k_1) \cdots b^*_{\alpha_n}(k_n)|0) \\
&= \sum_{a_1, \cdots \alpha_n}^{N} \int d^3k_1 \cdots \int d^3k_n f_1^{\alpha_1}(k_1) \cdots f_n^{\alpha_n}(k_n) \cdot \sigma(\alpha_1, k_1; \ldots; \alpha_n, k_n) \\
&\quad \times [\sigma(\alpha_1, k_1; \ldots; \alpha_n, k_n) b^*_{\alpha_1}(k_1) \cdots b^*_{\alpha_n}(k_n)]|0) \\
&= \sum_{\alpha_1, \cdots \alpha_n}^{N} \int d^3k_1 \cdots \int d^3k_n f_1^{\alpha_1}(k_1) \cdots f_n^{\alpha_n}(k_n) \cdot \sigma(\alpha_n, k_1; \ldots; \alpha_n, k_n) \\
&\quad \times a^*_{\alpha_1}(k_1) \cdots a^*_{\alpha_n}(k_n)|0)
\end{aligned} \tag{9}$$

In the above formula we made use of the fact that the contributions from sets $(k_1, \ldots, k_n)$ on which the symbol σ vanishes are of Riemann-Lebesgue measure zero (Garbeczewski, 1985a; Hudson and Parthasarathy, 1986); hence we could safely insert $\sigma^2 = \sigma \cdot \sigma$. The antisymmetric n-point function

$$f_1^{\alpha_1}(k_1) \cdots f_n^{\alpha_n}(k_n) \cdot \sigma(\alpha_1, k_1; \ldots; \alpha_n, k_n) = f_{\text{antisym}}^{\alpha_1 \cdots \alpha_n}(k_1, \ldots, k_n)$$

upon multiplying by σ, is turned into a symmetric function which respects the Pauli principle [i.e., vanishes if any pair of (α, k) labels appears more than once], and conversely. For general Fock space vectors the following formula holds:

$$|f) \in \mathscr{H}$$

$$\begin{aligned} |f) = \sum_{n=0}^{\infty} \sum_{\alpha_1, \cdots \alpha_n} \sum^N \int d^3k_1 \cdots \int d^3k_n f^{\alpha_1, \ldots, \alpha_n}(k_1, \ldots, k_n) \\ \times a_{\alpha_1}^*(k_1) \cdots a_{\alpha_n}^*(k_n)|0) \\ = \sum_{n=0}^{\infty} \sum_{\alpha_1, \ldots, \alpha_n}^N \int d^3k_1 \cdots \int d^3k_n \\ \times [f^{\alpha_1, \ldots, \alpha_n}(k_1, \ldots, k_n)\sigma(\alpha_1, k_1; \ldots; \alpha_n, k_n)] \\ \times b_{\alpha_n}^*(k_1) \cdots b_{\alpha_n}^*(k_n)|0) \end{aligned} \tag{10}$$

and it is an easy exercise to verify that the (diagonal) operators

$$\begin{aligned} H_{\mathrm{F}} = \sum_{\alpha=1}^{N} \int d^3k\, E_k a_\alpha^*(k) a_\alpha(k) \\ H_{\mathrm{B}} = \sum_{\alpha=1}^{N} \int d^3k\, E_k b_\alpha^*(k) b_\alpha(k) \end{aligned} \tag{11}$$

satisfy the identities

$$\begin{aligned} H_{\mathrm{F}}|f) = H_{\mathrm{B}}|f) \qquad \forall |f) \in \mathscr{D} \subset \mathscr{H} \\ \exp(iH_{\mathrm{F}}t)|f) = \exp(iH_{\mathrm{B}}t)|f) \qquad \forall |f) \in \mathscr{H} \end{aligned} \tag{12}$$

[compare, e.g., (3)].

In connection with the formulas (12) and (3), which imply the identification $H_{\mathrm{B}} = H_{\mathrm{F}}$ on any domain that is chosen in common for the Bose and Fermi problem Fock space, one may raise the issue of whether the two Hamiltonians H_{B} and H_{F} do indeed describe the same physics. The experienced statistical physicist would immediately object against the identification of boson and fermion free energies. Moreover, the correlation functions that are used to determine the model are obviously different, depending on the choice of the boson or fermion statistics. Even worse, there exists an interesting example of the boson-fermion relationship (Schick, 1968) in which the free fermion Hamiltonian (energies of the form $p^2/2m$) is replaced by the boson Hamiltonian, which in addition to the bilinear Tomonaga term involves a trilinear interaction piece, to be viewed as a small perturbation.

All this would seemingly contradict our previous analysis. However, the situation is not that bad:

1. The identities (3) and (12) pertain to models with continuous (or almost continuous) energy-momentum spectrum, which is not the case in standard statistical physics considerations. A good example is just the model studied in Schick (1968), where the number N of fermions with spin $\frac{1}{2}$ is placed on a ring of finite radius, so that a discrete set of modes arises.

2. As usual with the boson description of Fermi systems (irrespective of whether we use a fermion pairing idea or not) on a momentum or configuration space lattice, the replacement of the fermion by the boson Hamiltonian amounts to a certain approximation scheme: it is meaningful only on a restricted set of states. These restrictions can be imposed to select a subset of Fermi states of the boson system (Garbaczewski, 1985b). But it may as well happen, and it does in Schick (1968), that the two Hamiltonians are put into equivalence only on a restricted set S of states of the Fermi system itself: the boson representation of the fermion Hamiltonian is then confined exclusively to S.

3. Though we were interested in the continuum mode distribution, there is an apparent link between the momentum space lattice studies of Luther and Schotte (1984) and Schick (1968) and ours. Namely, we can always pass to the variety of lattice approximations of the continuous system once given [Schick's (1968) model would be one of them] Provided the continuum limit can be approached smoothly, the differences between distinct lattice approximants can be disregarded in the appropriate (large site or mode number in the finite volume) regime.

We now give a more detailed description of point 3. Let the free boson field theory model be given in $1+1$ space-time dimensions: for concreteness, one may take the $c=0$ case of the nonlinear Schrödinger field (Garbaczewski, 1985b). Then the canonical generators

$$[B(k), B^*(p)]_- = \delta(p-k), \qquad B(k)|0) = 0 \quad \forall k; \quad k, p \in R$$

induce a sequence of lattice approximants:

$$b_s = \Delta^{-1/2} \int \chi_s(k) B(k)\, dk \tag{13}$$

where $\chi_s(k)$ is a characteristic function of the set $\Delta_s \in R$ whose length equals

Δ. We denote by $\mathscr{H}_B$ the boson Fock space. We introduce a projection P in $\mathscr{H}_B$:

$$P = \prod_s p_s$$
$$p_s =: \exp(-b_s^* b_s) : + b_s^* : \exp(-b_s^* b_s) : b_s \tag{14}$$

We observe that

$$H_B = \sum_k \varepsilon_k b_k^* b_k \to P H_B P = \sum_k \varepsilon_k \sigma_k^+ \sigma_k^- = H_F = \sum_k \varepsilon_k a_k^* a_k \tag{15}$$

where

$$Pb_s^* P = \sigma_s^+, \qquad Pb_s P = \sigma_s^-$$
$$[\sigma_s^{\neq}, \sigma_t^{\neq}]_- = 0, \quad s \neq t, \qquad [\sigma_s^-, \sigma_s^+]_+ = P$$
$$a_s^* = \sigma_s^+ \prod_{j<s} (1 - 2\sigma_j^+ \sigma_j^-) \tag{16}$$
$$[a_s, a_t^*]_+ = \delta_{st} P$$

Consequently, the spin-$\frac{1}{2}$ Pauli operators and the corresponding (via Jordan's map) Fermi ones can be consistently introduced on the proper subspace $P\mathscr{H}_B = \mathscr{H}_F$ of $\mathscr{H}_B$. Here

$$a_{j_1}^* \cdots a_{j_s}^* |0) = \varepsilon_{j_1 \cdots j_s} \sigma_{j_1}^+ \cdots \sigma_{j_s}^+ |0)$$
$$= \varepsilon_{j_1 \cdots j_s} b_{j_1}^* \cdots b_{j_s}^* |0) \tag{17}$$

For sequences of indices with noncoinciding entries, $\varepsilon_{j_1 \cdots j_s}$ is the completely antisymmetric Levi-Civita symbol, which takes values 0 or ± 1.

If we notice that

$$P = \prod_s p_s = :\exp\left(-\sum_s b_s^* b_s \right): + \sum_s b_s^* :\exp\left(-\sum_t b_t^* b_t \right): b_s$$
$$+ \frac{1}{2!} \sum_{s \neq t} b_s^* b_t^* :\exp\left(-\sum_r b_r^* b_r \right): b_t b_s + \cdots \tag{18}$$

then the formal passage from Δ to 0 (see, e.g., Garbaczewski, 1985a) gives

$$P = P_\Delta \to :\exp\left[-\int dk\, B^*(k) B(k) \right]:$$
$$+ \int dk\, B^*(k) :\exp\left[-\int dp\, B^*(p) B(p) \right]: B(k) + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int dk_1 \cdots \int dk_n\, B^*(k_1) \cdots B^*(k_n) [\sigma(k_1, \ldots, k_n)]^2$$
$$\times :\exp\left[-\int dp\, B^*(p) B(p) \right]: B(k_1) \cdots B(k_n) = 1_F \tag{19}$$

where 1_F is a continuum field theory operator (the operator unit of the Fock representation of the CAR algebra in the boson Fock space) known from our earlier studies of relationships between fermions and bosons (Garbaczewski, 1985b).

In the above, $\sigma(k_1, \ldots, k_n)$ is a completely antisymmetric symbol, a good example being provided by

$$\sigma(k_1, \ldots, k_n) = \prod_{1 \le j < i \le n} [\Theta(k_i - k_j) - \Theta(k_j - k_i)] \tag{20}$$

where $\Theta(k)$ is the Heaviside function.

The relevant feature of 1_F is that although it is a projection in the boson Fock space $\mathcal{H}_B$, it is in fact trivial, and can be identified with the operator unit 1_B of the boson algebra, so that

$$P\mathcal{H}_B = \mathcal{H}_F \subset \mathcal{H}_B \xrightarrow[\Delta \to 0]{} 1_F \mathcal{H}_B = \mathcal{H}_F = \mathcal{H}_B \tag{21}$$

which implies the unification of the boson and fermion Fock space (Garbaczewski, 1985a; Hudson and Parthasarathy (1986). Namely, the boson Fock space is spanned by vectors

$$\begin{aligned} |f) = \sum_n \frac{1}{(n!)^{1/2}} \int dk_1 \cdots \int dk_n f(k_1, \ldots, k_n) \\ \times B^*(k_1) \cdots B^*(k_n)|0) \end{aligned} \tag{22}$$

If we take account of the fact that each n-point wave function can be decomposed as $f = \sigma^2 f + (1 - \sigma^2) f$, where the second term is different from zero on the set of Lebesgue measure zero in R^n, we find

$$\begin{aligned} |f = {} & \sum_n \frac{1}{(n!)^{1/2}} \int dk_1 \cdots \int dk_n \, \{f(k_1, \ldots, k_n)\sigma(k_1, \ldots, k_n)\} \\ & \times \sigma(k_1, \ldots, k_n) B^*(k_1) \cdots B^*(k_n)|0) \\ = {} & \sum_n \frac{1}{(n!)^{1/2}} \int dk_1 \cdots \int dk_n \, \{f(k_1, \ldots, k_n)\sigma(k_1, \ldots, k_n)\} \\ & \times A^*(k_1) \cdots A^*(k_n)|0) \end{aligned} \tag{23}$$

$$[A(k), A^*(p)]_+ = \delta(k - p)$$

$$A^*(k)|0 = B^*(k)|0), \qquad A(k)|0) = 0 \quad \forall k \in R$$

In the above the Fermi generators $\{A^*(k), A(k)\}$ were introduced, which is allowed by Garbaczewski and Rzewuski (1974) and Garbaczewski (1975) (see also Garbaczewski, 1985a, b): the same state vector thus describes the boson and fermion cases in $\mathcal{H}_B$. The symmetric or antisymmetric wave functions are merely the different coordinate representatives of this vector, depending on the particular choice of the boson or fermion basis in $\mathcal{H}_B$. When Δ goes to zero, the proper subspace $P\mathcal{H}_B = \mathcal{H}_F$ of $\mathcal{H}_B$ in fact extends to the whole of $\mathcal{H}_B$ (one may as well say that $\mathcal{H}_B$ collapses to $\mathcal{H}_F$) and differences between distinct boson versions of the free fermion model on a lattice can be disregarded in this regime.

3. REVIEW OF SYMMETRY PROPERTIES: CONSISTENCY CHECKS

We begin by stating that all the basic notations, including conventions for Dirac gamma matrices, are standard (Bjorken and Drell, 1965; Itzykson and Zuber, 1980; Lee, 1981) in the present paper. In Luther and Schotte (1984) nonstandard conventions were used, which makes a difference if we turn to explicit transformation formulas.

Compared with Weinberg (1964a–c, 1965) and Lee (1981) we make one departure, by making use only of the explicit helicity parametrization $s = \pm 1$ with spin value considered separately (Weinberg and Lee use $\pm\frac{1}{2}$ or ± 1 labeling). Let us introduce the following notation for the boson operators related to the four-component neutrino problem:

$$\begin{aligned} D^*(p) &= \begin{pmatrix} D^*(p,+) \\ D^*(p,-) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha_1^* + i\alpha_2^* \\ \alpha_1^* - i\alpha_2^* \end{pmatrix}(p) \\ D(p) &= \begin{pmatrix} D(p,+) \\ D(p,-) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 \end{pmatrix}(p) \\ B^*(p) &= \begin{pmatrix} B^*(p,+) \\ B^*(p,-) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \beta_1^* + i\beta_2^* \\ \beta_1^* - i\beta_2^* \end{pmatrix}(p) \\ B(p) &= \begin{pmatrix} B(p,+) \\ B(p,-) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \beta_1 - i\beta_2 \\ \beta_1 + i\beta_2 \end{pmatrix}(p) \end{aligned} \tag{24}$$

where

$$\begin{gathered} [\alpha_i(p), \alpha_j^*(q)]_- = \delta_{ij}\delta(p-q) = [\beta_i(p), \beta_j^*(q)]_- \\ [\alpha_i^{\#}(p), \beta_j^{\#}(q)]_- = 0 \\ \alpha_i(p)|0) = \beta_i(p)|0) = 0, \qquad \forall i, p \end{gathered} \tag{25}$$

and the boson equivalent of the neutrino Hamiltonian reads

$$H_0 = \sum_{s=\pm} \int d^3p\, [B^*(p,s)B(p,s) + D^*(p,s)D(p,s)] \cdot |p| \tag{26}$$

Our problem now is to demonstrate that within the Hamiltonian formalism, when supplemented by the boson-fermion duality observation (3), H_0 describes both spin-$\frac{1}{2}$ (Fermi) and spin-1 (Bose) particles for the mass-zero case. The state (Fock) space is here common for bosons and fermions. For this purpose the knowledge of transformation properties and field constructions catalogued in Weinberg (1964a–c, 1965) is crucial.

To facilitate further discussion (we in fact anticipate the final conclusion), we shall admit that in the Hilbert space $\mathscr{H}$ of the four-component neutrino problem, in addition to the neutrino fields, we can consistently define two independent four-potentials (not fields in Weinberg's terminology), which are quantized according to Weinberg's strictly physical (helicity) procedure (see Weinberg 1965; Strocchi, 1970).

1. The Hilbert space is equipped with a positive-definite metric, and only the physical (helicity plus or minus) states are allowed to arise in it. Then the analysis of the Poincaré group representations follows the standard pattern. The Fock representation of the CCR is used.

2. The Lorentz condition $\partial_\mu A^\mu = 0$ is imposed on the potential A_μ as an operator identity. We assume the same with respect to B_μ.

3. The fourth component A_0 of the potential is required to be identical to zero (the same for B_0), which results in the manifest Coulomb gauge.

As is well known, in this formulation potentials lack manifest covariance and do not satisfy weak local commutativity, which is, however, the price to be paid if the purely physical state space is admitted. In the Gupta–Bleuler formalism the set of physical states cannot be dense in the (indefinite metric) Hilbert space.

According to Weinberg (1965), $m=0$, spin 1 theory allows for two possible choices of the potentials for the Maxwell framework. They differ with respect to parity and time reversal. For the normal "electric" potential A_μ we have

$$\begin{aligned} PA^\mu(x)P^{-1} &= -A^\mu(-x, t) \\ TA^\mu(x)T^{-1} &= -A^\mu(x, -t) \end{aligned} \tag{27}$$

while for the abnormal "magnetic" potential

$$\begin{aligned} PB^\mu(x)P^{-1} &= B^\mu(-x, t) \\ TB^\mu(x)P^{-1} &= B^\mu(x, -t) \end{aligned} \tag{28}$$

The magnetic potential is known not to couple charged currents. For charge conjugation we have as usual $CA^\mu(x)C^{-1} = -A^\mu(x)$ and $CB^\mu(x)C^{-1} = -B^\mu(x)$, so that the CPT transformation of both potentials is the same.

We adopt the following definition (β^* and β's arise if we replace A^μ by B^μ):

$$\begin{aligned} \alpha_\lambda(k) &= \int \frac{d^3x \exp(ikx)}{[2|k|(2\pi)^3]^{1/2}} \boldsymbol{\varepsilon}(k, \lambda) \cdot [|k|\mathbf{A}(x) + i\dot{\mathbf{A}}(x)] \\ \alpha_\lambda^*(k) &= \int \frac{d^3x \exp(-ikx)}{[2|k|(2\pi)^3]^{1/2}} \boldsymbol{\varepsilon}(k, \lambda) \cdot [|k|\mathbf{A}(x) - i\dot{\mathbf{A}}(x)] \end{aligned} \qquad \lambda = 1, 2 \tag{29}$$

which, upon adopting the convention

$$\varepsilon(-k, 1) = \varepsilon(k, 1), \qquad \varepsilon(-k, 2) = -\varepsilon(k, 2) \tag{30}$$

implies that the operators for the circularly polarized radiation

$$\begin{aligned}
\alpha^*(k, +) &= \frac{1}{\sqrt{2}}[\alpha_1^*(k) + i\alpha_2^*(k)] \doteq D^*(k, +)\\
\alpha^*(k, -) &= \frac{1}{\sqrt{2}}[\alpha_1^*(k) - i\alpha_2^*(k)] \doteq D^*(k, -)\\
\beta^*(k, +) &= \frac{1}{\sqrt{2}}[\beta_1^*(k) + i\beta_2^*(k)] \doteq B^*(k, +)\\
\beta^*(k, -) &= \frac{1}{\sqrt{2}}[\beta_1^*(k) - i\beta_2^*(k)] \doteq B^*(k, -)
\end{aligned} \tag{31}$$

transform according to

$$\begin{aligned}
P\alpha^{\#}(p, s)P^{-1} &= -\alpha^{\#}(-p, -s)\\
P\beta^{\#}(p, s)P^{-1} &= \beta^{\#}(-p, -s)\\
T\alpha^{\#}(p, s)T^{-1} &= -\alpha^{\#}(-p, s)\\
T\beta^{\#}(p, s)T^{-1} &= \beta^{\#}(-p, s)
\end{aligned} \tag{32}$$

while

$$\begin{aligned}
C\alpha^{\#}(p, s)C^{-1} &= -\alpha^{\#}(p, s)\\
C\beta^{\#}(p, s)C^{-1} &= -\beta^{\#}(p, s)
\end{aligned} \tag{33}$$

Obviously, C, P, and T are symmetries of the Hamiltonian (26) irrespective of whether we relate operators B^*, B, D^*, and D to spin-$\frac{1}{2}$ or spin-1 particles,

$$\begin{aligned}
[H_0, C_1] = 0 = [H_0, P_1]_- &= [H_0, T_1]_-\\
[H_0, C_{1/2}]_- = 0 = [H_0, P_{1/2}]_- &= [H_0, T_{1/2}]_-
\end{aligned} \tag{34}$$

The subscript 1 or $\frac{1}{2}$ refers to the choice of the transformation properties for the same operators B^*, B, D^*, and D.

For our later purposes we observe that H_0 also displays another invariance connected with the map

$$A^\mu(x) \to \pm B^\mu(x) \tag{35}$$

which seems to be rooted in the dyality (duality) symmetry of electrodynamics with magnetic charges (Cabibbo and Ferrari, 1962; Rohrlich, 1966; Han and Biedenharn, 1971).

Though it is an academic exercise to check that standard symmetries of the spin-$\frac{1}{2}$ mass-zero (in fact, discrete symmetries display the same property when the mass of the Fermi field is nonzero) obey (34), it is, however, worth indicating how the spin-$\frac{1}{2}$ and spin-1 transformations of the same operators are related. Namely (spin-$\frac{1}{2}$ label refers to spin-$\frac{1}{2}$ transformation rules) we have

$$\begin{aligned} P_{1/2}D^*(p,s)P_{1/2}^{-1} &= -\eta_P D^*(-p,-s) = \eta_P P_1 D^*(p,s)P_1^{-1} \\ P_{1/2}D(p,s)P_{1/2}^{-1} &= -\eta_P^* D(-p,-s) = \eta_P^* P_1 D(p,s)P_1^{-1} \\ P_{1/2}B^*(p,s)P_{1/2}^{-1} &= \eta_P^* B^*(-p,-s) = \eta_P^* P_1 B^*(p,s)P_1^{-1} \\ P_{1/2}B(p,s)P_{1/2}^{-1} &= \eta_P B(-p,-s) = \eta_P P_1 B(p,s)P_1^{-1} \end{aligned} \tag{36}$$

which differs by a phase factor $|\eta_P| = 1$ from the spin-1 transformation formulas.

Analogously for the time reversal we have

$$\begin{aligned} T_{1/2}D^*(p,s)T_{1/2}^{-1} &= [-\exp i\Theta(-p,s)]T_1 D^*(p,s)T_1^{-1} \\ T_{1/2}D(p,s)T_{1/2}^{-1} &= \{-\exp[-i\Theta(-p,s)]\}T_1 D(p,s)T_1^{-1} \\ T_{1/2}B^*(p,s)T_{1/2}^{-1} &= [\exp i\Theta(p,s)]T_1 B^*(p,s)T_1^{-1} \\ T_{1/2}B(p,s)T_{1/2}^{-1} &= \{\exp[-i\Theta(p,s)]\}T_1 B(p,s)T_1^{-1} \end{aligned} \tag{37}$$

Here $\exp[-i\Theta(p,s)+i\Theta(-p,s)] = 1$.

With respect to the charge conjugation we have

$$\begin{aligned} C_{1/2}B(p,s)C_{1/2}^{-1} &= \eta_c D(p,s) \\ C_{1/2}B^*(p,s)C_{1/2}^{-1} &= \eta_C^* D^*(p,s) \\ C_{1/2}D(p,s)C_{1/2}^{-1} &= \eta_C^* B(p,s) \\ C_{1/2}D^*(p,s)C_{1/2}^{-1} &= \eta_C B^*(p,s) \end{aligned} \tag{38}$$

which means that up to the phase factors, spin-1/2 transformations are generated by spin-1 transformations combined with the duality mapping of potentials $A^\mu \to \pm B^\mu$.

Conventionally the real distinction between spin 1 and spin $\frac{1}{2}$ is attributed to the explicit spin factor which enters the transformation formulas with respect to rotations. Indeed, there is merely a phase difference (as in the case of discrete transformations), since

$$\begin{aligned} U_{1/2}(\Lambda)B^*(p,s)U_{1/2}^{-1}(\Lambda) &= (|\Lambda p|/|p|)^{1/2} \\ &\quad \times \exp[\tfrac{1}{2}is\Theta(p,\Lambda)]\cdot B^*(\Lambda p,s) \\ U_{1/2}(\Lambda)B(p,s)U_{1/2}^{-1}(\Lambda) &= (|\Lambda p|/|p|)^{1/2} \\ &\quad \times \exp[-\tfrac{1}{2}is\Theta(p,\lambda)]\cdot B(\Lambda p,s) \end{aligned} \tag{39}$$

and $D^*(p, -s)$ transforms like $B(p, s)$, while $D(p, -s)$ transforms like $B^*(p, s)$:

$$\begin{aligned} &U_{1/2}(\Lambda) D^*(p, -s) U_{1/2}^{-1}(\Lambda) \\ &\quad = (|\Lambda p|/|p|)^{1/2} \exp[-\tfrac{1}{2} i s \Theta(p, \Lambda)] \cdot D^*(\Lambda p, -s) \end{aligned} \tag{40}$$

Passage to spin-1 transformation rules amounts to keeping the same transformation rule (39) with $s = \pm 1$ replaced by $2s$ for both the electric and magnetic cases.

It is also useful to know that

$$\begin{aligned} H_0^\Lambda &= U_{1/2}(\Lambda) H_0 U_{1/2}^{-1}(\Lambda) \\ &= U_1(\Lambda) H_0 U_1^{-1}(\Lambda) \\ &= \sum_{s=\pm} \int d^3p\, |\Lambda p| [B^*(\Lambda p, s) B(\Lambda p, s) + D^*(\Lambda p, s) D(\Lambda p, s)] \end{aligned} \tag{41}$$

i.e., $[H_0, H_0^\Lambda]_- = 0$. We thus have an explicit proof that apart from the equivalence of Hamiltonians, the transformation properties of spin-1 and spin-$\frac{1}{2}$ type lead to outcomes differing in the phases. However, it is not our aim (in contrast with what the standard neutrino theory of light demands) to get the same transformation properties for photon relatives of neutrinos. The general Weinberg analysis demonstrates in fact that from the same set of operators we can construct massless fields corresponding either to spin 1 or spin $\frac{1}{2}$, which reside in a common Hilbert space, and whose dynamics is governed by the same (Bose–Fermi equivalence must be invoked at this point) Hamiltonian. This means finally that our Hamiltonian (26) has in fact two formally distinct but in all respects equivalent realizations: in terms of the relativistic spin-$\frac{1}{2}$ field describing the mass-zero four-component neutrino, and in terms of the doublet of electric and magnetic Maxwell potentials, which in our case are introduced as independent operator quantities, $[A^\mu(x), B^\nu(y)]_- = 0$.

4. CONCLUSIONS

1. The Fock space $\mathcal{H}$ for the four-component neutrino and the two (magnetic and electric) species of Maxwell field is the same. The Hamiltonian H_0 of (26) is common (via Bose–Fermi duality) for both systems, provided we identify boson operators $B^\#(p, \pm), D^\#(p, \pm)$ with operators $\beta^\#(p, \pm)$, $\alpha^\#(p, \pm)$ for circularly polarized radiation (magnetic and electric, respectively).

2. The corresponding Hamiltonian simultaneously governs the dynamics of two formally distinct fields (which can be constructed in $\mathcal{H}$

following the Weinberg recipies), the spin-1 and spin-$\frac{1}{2}$ fields, which, though displaying different transformation properties, have precisely the same physical content.

3. In accordance with Weinberg's rules to construct relativistic fields for the case of spin $\frac{1}{2}$, $m=0$, the left- and right-handed neutrino operator $b(p,\pm)$ and (or) $B(p,\pm)$ refers to the left- and right-handed "magnetic" photon, which does not couple to charged currents. The antineutrino corresponds here to the standard electric photon.

4. It is incorrect to say that neutrinos are constructed from photons or photons from neutrinos. As massless particles in the above framework they are in fact identical.

5. What is the deeper physical meaning (if any) of the above results we are not competetent to debate. The same concerns the existence or nonexistence of the right-handed neutrino (Barut, 1984). The most appealing feature in the above is the fact that the magnetic potential B^{μ} plays a distinguished role and allows for the identification of the "magnetic" photon (so far never introduced) with the neutrino. The fact that it does not couple to charged currents is very attractive in this context.

6. In our second-quantized approach two independent operator-valued potentials arise, and there is no way to express one potential in terms of another. If it would have been possible, we would have reduced the problem to the first-quantized level, when the potentials are simply classical objects. The potentials can then be related, which is the case in the tensorial description of neutrinos (Penney, 1965) or in constructing the spinor representation of the Maxwell equations (Moses, 1968; Takahashi and Okuda, 1983). An analogous situation is encountered in Aratyn (1983), where one departs from the antisymmetric tensor fields.

ACKNOWLEDGMENTS

I would like to thank Profs. S. Albeverio, Ph. Blanchard, and L. Streit for an invitation to BiBoS Bielefeld, where the present paper was written. I am grateful to Prof. R. F. Streater for discussions concerning the boson-fermion Fock space unification by Hudson and Parthasarathy. Thanks to Prof. A. Luther I learned about Schick's (1968) paper, which strongly affected the final presentation of the topics in the present investigation.

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